J Math Chem (2011) 49:1646–1666
DOI 10.1007/s10910-011-9848-x

ORIGINAL PAPER

Delay induced oscillations in a turbidostat with feedback control

Sanling Yuan · Pan Li · Yongli Song

Received: 5 October 2010 / Accepted: 16 May 2011 / Published online: 1 June 2011

Abstract A model of competition between two species in a turbidostat with delayed feedback control is investigated. By choosing the delay in the measurement of the optical sensor to the turbidity of the fluid as a bifurcation parameter, we show that Hopf bifurcations can occur as the delay crosses some critical values. The direction and stability of the bifurcating periodic solutions are determined by the normal form theory and the center manifold theorem. Computer simulations illustrate the results.

Keywords Turbidostat · Time delay · Hopf bifurcation · Stability · Periodic solution

1 Introduction

The chemostat, a laboratory apparatus used for the continuous culture of microorganisms, has played an important role in microbiology and population biology (see [1–5], for example). It is the most simple idealization of a biological system where the parameters are measurable, the experiments are reasonable, and the mathematics is tractable [4]. A classical result which is known mathematically [6–14] and experimentally [15–17] is the competitive exclusion principle, i.e., only one organism survives in the long run while others die out. There is a large literature devoted to the

S. Yuan (✉) · P. Li
College of Science, Shanghai University for Science and Technology, Shanghai 200093, China
e-mail: sanling@usst.edu.cn

P. Li
e-mail: lipan3001@163.com

Y. Song
Department of Mathematics, Tongji University, Shanghai 200092, China
e-mail: syl.mail@163.com

modifying the chemostat model to ensure the coexistence of the organisms (see, for example, [21,28–38] and also [4] for a review). Recently, much attention has been devoted to the modification of the chemostat to ensure the coexistence of species on a single nutrient by controlling the dilution rate of the chemostat [15,17–20,28].

The chemostat with feedback control of the dilution rate is often referred to as turbidostat by bio-engineers and biologists (see Panikov [22] and Shuler and Kargi [23]. In the turbidostat, an optical sensor measures the turbidity of the fluid and this signal is used to control the dilution rate. Coexistence of two species in the turbidostat was shown numerically by Flegr [19], and later analytically by De Leenheer and Smith [18]. Their model takes the following form:

$$\begin{aligned} \dot{S} &= D(x)(S^0 - S) - \frac{x_1}{\gamma_1} f_1(S) - \frac{x_2}{\gamma_2} f_2(S), \\ \dot{x}_1 &= x_1(f_1(S) - D(x)), \\ \dot{x}_2 &= x_2(f_2(S) - D(x)), \end{aligned} \tag{1.1}$$

where $S(t)$ is the limiting nutrient concentration and $x_i(t)(i = 1, 2)$ is the concentration of the ith competitor at time t. S^0 is the input concentration of the limiting nutrient and γ_i are yield constants. The dilution of the turbidostat is $D(x)$ that is controlled by setting

$$D(x) = d + k_1 x_1(t) + k_2 x_2(t), \tag{1.2}$$

where $d > 0$, $k_i > 0$ for $i = 1, 2$. The functions f_i are called uptakes functions and each f_i is assumed to be a continuously differentiable function with $f_i(0) = 0$ and $f_i' > 0$ for all $S \in \mathbb{R}_+$. A typical form of the f_i are Monod functions

$$f_i(S) = \frac{m_i S}{a_i + S}, \quad i = 1, 2, \tag{1.3}$$

where $m_i, a_i (i = 1, 2)$ are, respectively, the maximal growth rate and the half-saturation constant or Michaelis-Menten constant of the ith competitor.

It is shown in [18] that model (1.1) permits a unique coexistence equilibrium for certain parameter values, and if the dilution rate depends affinely on the concentrations of two competing organisms, coexistence may be achieved as a globally asymptotically stable equilibrium point in the interior of the non-negative orthant. This implies that if accurate measurements of the concentrations of the competing organisms are available, a simple affine control algorithm would allow automated coexistence in the chemostat (see [18] for the details). But in reality, no matter how sensitive the sensor is there always exists delay in the process of its measurement to the turbidity of the fluid and this signal is used to control the dilution rate. For a long time, it has been recognized that delays can have very complicated impact on the dynamics of a system (see, for example, [24–26]). For example, delays can cause the loss of stability and can induce various oscillations and periodic solutions. In the present paper, we investigate the effect of delay on the dynamics of model (1.1) with the dilution rate $D(x)$ given by (1.2). Taking the delay as a parameter, our results show that Hopf bifurcations occur as delay passes through some critical values, i.e., a family of periodic solutions will be bifurcated from the positive equilibrium. Furthermore, the formulae determining the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions are given. Numerical simulations are carried out to illustrate our results.

This paper is organized as follows. The model and some basic results about the model are presented in the next section. In Sect. 3, we consider the stability and the local Hopf bifurcation of the positive equilibrium. In Sect. 4, we employ the normal form method and the center manifold theory introduced by Hassard et al. [27] to analyze the direction, stability and the period of the bifurcating periodic solution at the critical values of the delay. In Sect. 5 we give a numerical example to illustrate our theoretical results in Sect. 4. Finally, a brief discussion is presented in Sect. 6.

2 The model

Let $S(t), x_1(t), x_2(t)$ have the similar biological meaning as in model (1.1). Consider the following model of exploitative competition for a nutrient between two species of microorganisms in a turbidostat with delayed feedback control:

$$\begin{aligned}
\dot{S} &= (d + k_1x_1(t-\tau) + k_2x_2(t-\tau))(S^0 - S) - \frac{x_1}{\gamma_1}f_1(S) - \frac{x_2}{\gamma_2}f_2(S),\\
\dot{x}_1 &= x_1[f_1(S) - (d + k_1x_1(t-\tau) + k_2x_2(t-\tau))],\\
\dot{x}_2 &= x_2[f_2(S) - (d + k_1x_1(t-\tau) + k_2x_2(t-\tau))]
\end{aligned} \tag{2.1}$$

with initial value conditions

$$S(\theta) = \varphi_0(\theta), \quad x_i(\theta) = \varphi_i(\theta), \quad i = 1, 2, \quad \theta \in (-\tau, 0], \tag{2.2}$$

where $\varphi_i(\theta) \in BC[-\tau, 0],\ i = 0, 1, 2$, the Banach space of all continuous bounded functions, and all parameters are positive constants. $S^0, \gamma_i, d, k_i, f_i(S)$ play similar roles as in model (1.1). τ is the delay of the optical sensor in the measurement of the turbidity of the fluid and $d + k_1x_1(t-\tau) + k_2x_2(t-\tau)$ is the dilution of the turbidostat that is controlled. When $\tau = 0$, the model becomes model (1.1) considered by Leenheer and Smith in [18].

It is convenient to perform scaling for chemostat type problems. Let

$$\bar{S} = \frac{S}{S^0}, \quad \bar{x}_i = \frac{x_i}{\gamma_i S^0}, \quad \bar{k}_i = \gamma_i S^0 k_i, \quad \bar{f}_i(\bar{S}) = f_i(S^0\bar{S}).$$

After dropping the bars, the model (2.1) becomes

$$\begin{aligned}
\dot{S} &= (d + k_1x_1(t-\tau) + k_2x_2(t-\tau))(1 - S) - x_1f_1(S) - x_2f_2(S),\\
\dot{x}_1 &= x_1(f_1(S) - (d + k_1x_1(t-\tau) + k_2x_2(t-\tau))),\\
\dot{x}_2 &= x_2(f_2(S) - (d + k_1x_1(t-\tau) + k_2x_2(t-\tau))).
\end{aligned} \tag{2.3}$$

As in [18], we introduce the following standing hypothesis:

(**H**) The graphs of the functions f_1 and f_2 intersect once at S^*:

$$f_1(S^*) = f_2(S^*) = D^* \tag{2.4}$$

where $S^* \in (0, 1)$. Moreover $f_1'(S^*) \neq f_2'(S^*)$.

Assumption (H) shows that for low values of the dilution rate ($D < D^*$), one of the two organisms wins the competition and for higher values ($D > D^*$, but $D < D_{\max} = \max\{f_1(1), f_2(1)\}$), the other wins the competition.

Note that if $d \in (0, D^*)$ and

$$k_1 < \frac{D^* - d}{1 - S^*} < k_2 \quad \text{or} \quad k_2 < \frac{D^* - d}{1 - S^*} < k_1, \tag{2.5}$$

then system (2.3) possesses four equilibrium points:

$$E_0 = (1, 0, 0), \quad E_1 = (1 - \lambda_1, \lambda_1, 0), \quad E_2 = (1 - \lambda_2, 0, \lambda_2) \\ \text{and} \quad E^* = (S^*, x_1^*, x_2^*) \tag{2.6}$$

where λ_i, S^* and x_i^* are given by:

$$\begin{aligned} f_i(\lambda_i) &= k_i(1 - \lambda_i) + d, \quad i = 1, 2, \\ x_1^* &= \frac{D^* - k_2(1 - S^*) - d}{k_1 - k_2}, \\ x_2^* &= \frac{k_1(1 - S^*) - D^* + d}{k_1 - k_2}. \end{aligned} \tag{2.7}$$

When $\tau = 0$, we have known the following results from Leenheer and Smith [18]:

Lemma 2.1 *Assume* $\tau = 0$ *and* $d \in (0, D^*]$. *If* (2.5) *and*

$$(k_1 - k_2)(f_1'(S^*) - f_2'(S^*)) > 0 \tag{2.8}$$

hold, then the positive equilibrium E^* *is global asymptotical stable with respect to initial conditions in int*$(\mathbb{R}_+^3)$.

3 Stability of the positive equilibrium and Hopf bifurcations

Assuming that $d \in (0, D^*)$ and (2.5) hold, we have known from above that system (2.3) has a unique positive equilibrium $E^* = (S^*, x_1^*, x_2^*)$. In the following, we will consider the stability of E^* and Hopf bifurcations induced by delay.

Let $y_1(t) = S(t) - S^*$, $y_2(t) = x_1(t) - x_1^*$, $y_3(t) = x_2(t) - x_2^*$. System (2.3) becomes

$$\begin{aligned}
\dot{y}_1 &= -ay_1(t) - D^* y_2(t) - D^* y_3(t) + k_1(1 - S^*)y_2(t-\tau) + k_2(1 - S^*)y_3(t-\tau)\\
&\quad -(b_{12}x_1^* + b_{22}x_2^*)y_1^2(t) - b_{11}y_1(t)y_2(t) - b_{21}y_1(t)y_3(t) - k_1 y_1(t)y_2(t-\tau)\\
&\quad -k_2 y_1(t)y_3(t-\tau) - (b_{13}x_1^* + b_{23}x_2^*)y_1^3(t) - b_{12}y_1^2(t)y_2(t) - b_{22}y_1^2(t)y_3(t) + \cdots,\\
\dot{y}_2 &= b_{11}x_1^* y_1(t) - k_1 x_1^* y_2(t-\tau) - k_2 x_1^* y_3(t-\tau)\\
&\quad +b_{12}x_1^* y_1^2(t) + b_{11}y_1(t)y_2(t) - k_1 y_2(t)y_2(t-\tau) - k_2 y_2(t)y_3(t-\tau)\\
&\quad +b_{13}x_1^* y_1^3(t) + b_{12}y_1^2(t)y_2(t) + \cdots,\\
\dot{y}_3 &= b_{21}x_2^* y_1(t) - k_1 x_2^* y_2(t-\tau) - k_2 x_2^* y_3(t-\tau)\\
&\quad +b_{22}x_2^* y_1^2(t) + b_{21}y_1(t)y_3(t) - k_1 y_2(t-\tau)y_3(t) - k_2 y_3(t)y_3(t-\tau)\\
&\quad +b_{23}x_2^* y_1^3(t) + b_{22}y_1^2(t)y_3(t) + \cdots,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
a &= (d + k_1 x_1^* + k_2 x_2^*) + b_{11}x_1^* + b_{21}x_2^*\\
&= D^* + b_{11}x_1^* + b_{21}x_2^*,\\
b_{ij} &= \frac{f_i^{(j)}(S^*)}{j!}, \quad i = 1, 2, \quad j = 1, 2, 3.
\end{aligned} \tag{3.2}$$

We then obtain the linearized system

$$\begin{aligned}
\dot{z}_1 &= -az_1(t) - D^* z_2(t) - D^* z_3(t) + k_1(1 - S^*)z_2(t-\tau) + k_2(1 - S^*)z_3(t-\tau),\\
\dot{z}_2 &= b_{11}x_1^* z_1(t) - k_1 x_1^* z_2(t-\tau) - k_2 x_1^* z_3(t-\tau),\\
\dot{z}_3 &= b_{21}x_2^* z_1(t) - k_1 x_2^* z_2(t-\tau) - k_2 x_2^* z_3(t-\tau).
\end{aligned}$$

From the linearized system we obtain the characteristic equation

$$(\lambda + D^*)[(\lambda^2 + r\lambda) + (p\lambda + q)e^{-\lambda\tau}] = 0, \tag{3.3}$$

where $r = b_{11}x_1^* + b_{21}x_2^* (> 0)$, $p = k_1 x_1^* + k_2 x_2^* (> 0)$, and $q = (k_1 - k_2)(b_{21} - b_{11})x_1^* x_2^*$.

To study the stability of the equilibrium E^* and the Hopf bifurcation of (2.3), it is sufficient to study the distribution of roots of Eq. (3.3). Observing that $\lambda = -D^*$ is always a negative root of (3.3), we need only consider the distribution of roots of the following equation:

$$(\lambda^2 + r\lambda) + (p\lambda + q)e^{-\lambda\tau} = 0. \tag{3.4}$$

Obviously, $\lambda = 0$ is not a root of (3.4). For $\tau = 0$ the characteristic equation becomes

$$\lambda^2 + (p+r)\lambda + q = 0, \tag{3.5}$$

from which we immediately have:

Lemma 3.1 *When $\tau = 0$, we have:*

(1) *if $q < 0$, then Eq. (3.5) always has two real roots: one positive root and one negative root,*
(2) *if $q > 0$, then Eq. (3.5) always has two roots with negative real parts.*

Now for $\tau > 0$, if $\lambda = i\omega(\omega > 0)$ is a root of Eq. (3.4), then we have

$$-\omega^2 + r\omega i + (p\omega i + q)e^{-i\omega\tau} = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} \omega^2 = q\cos(\omega\tau) + p\omega\sin(\omega\tau), \\ r\omega = q\sin(\omega\tau) - p\omega\cos(\omega\tau), \end{cases} \tag{3.6}$$

which leads to the following fourth degree polynomial equation

$$\omega^4 + (r^2 - p^2)\omega^2 - q^2 = 0. \tag{3.7}$$

It is easy to see that (3.7) has only one positive real root

$$\omega_* = \sqrt{\frac{-(r^2-p^2) + \sqrt{(r^2-p^2)^2 + 4q^2}}{2}}. \tag{3.8}$$

We can now find the value of τ by substituting ω_* into (3.6) and solving for τ. After computing, we obtain

$$\tau_j = \frac{1}{\omega_*}\arccos\left(\frac{q-pr}{\omega_*^2 + r^2}\right) + \frac{2j\pi}{\omega_*}, \quad j = 0, 1, 2, \ldots. \tag{3.9}$$

Thus, when $\tau = \tau_j$, the characteristic equation (3.4) has a pair of purely imaginary roots $\pm i\omega_*$.

Denote

$$\lambda_j(\tau) = \eta_j(\tau) + i\omega_j(\tau)$$

the root of Eq. (3.4) near $\tau = \tau_j$ satisfying $\eta_j(\tau_j) = 0$ and $\omega_j(\tau_j) = \omega_*$.

Lemma 3.2 *The following transversality condition is satisfied:*

$$\left.\frac{d\eta_j(\tau)}{d\tau}\right|_{\tau=\tau_j} > 0.$$

Proof Differentiating Eq. (3.4) with respect to τ we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + r)e^{\lambda\tau}}{\lambda(p\lambda + q)} + \frac{p}{\lambda(p\lambda + q)} - \frac{\tau}{\lambda} \tag{3.10}$$

which, together with (3.4), (3.7) and (3.8), leads to

$$\begin{aligned}
sign\left\{\frac{d(Re(\lambda))}{d\tau}\right\}_{\tau=\tau_j} &= sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_j} \\
&= sign\left\{Re\left[\frac{(2\lambda + r)e^{\lambda\tau}}{\lambda(p\lambda + q)}\right]_{\tau=\tau_j} + Re\left[\frac{p}{\lambda(p\lambda + q)}\right]_{\tau=\tau_j}\right\} \\
&= sign\left\{Re\left[\frac{-(2\lambda + r)}{\lambda^2(\lambda + r)}\right]_{\tau=\tau_j} - \frac{p^2}{p^2\omega_*^2 + q^2}\right\} \\
&= sign\left\{\frac{r^2 + 2\omega_*^2}{\omega_*^2(r^2 + \omega_*^2)} - \frac{p^2}{p^2\omega_*^2 + q^2}\right\} \\
&= sign\{r^2 + 2\omega_*^2 - p^2\} \\
&= sign\left\{\sqrt{(r^2 - p^2)^2 + 4q^2}\right\} = 1.
\end{aligned}$$

The proof is thus completed. □

From Lemmas 3.1 and 3.2, we can obtain the following results about the distribution of the characteristic roots of Eq. (3.3).

Lemma 3.3 *Suppose that ω_* and $\tau_j (j = 0, 1, 2, \ldots)$ are defined by (3.8) and (3.9), respectively.*

(1) *if $q < 0$, then Eq. (3.3) has at least one root with positive real part for all $\tau > 0$;*
(2) *if $q > 0$ and $\tau = \tau_j$, then Eq. (3.3) has a pair of simple imaginary roots $\pm i\omega_*$. Furthermore, if $\tau \in [0, \tau_0)$, then all roots of Eq. (3.3) have negative real parts; if $\tau = \tau_0$, then all roots of Eq. (3.3) except $\pm i\omega_*$ have negative real parts; if $\tau \in (\tau_j, \tau_{j+1})$, then Eq. (3.3) has $2(j + 1)$ roots with positive real parts.*

From Lemma 2.3, we easily obtain the following results about the stability of the positive equilibrium E^* and existence of Hopf bifurcation of (2.3).

Theorem 3.1 *Let ω_* and $\tau_j (j = 0, 1, 2, \ldots)$ are defined by (3.8) and (3.9), respectively.*

(1) *if $q < 0$, then the equilibrium E^* of the system (2.3) is unstable for all $\tau \geq 0$;*
(2) *if $q > 0$, then*
(a) *the equilibrium E^* of the system (2.3) is stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$;*
(b) *Hopf bifurcation occurs when $\tau = \tau_j$; that is, a family of periodic solutions bifurcate from the positive equilibrium E^* of system (2.3) as τ passes through the critical value τ_j.*

4 Direction and stability of the Hopf bifurcation solution

In the previous section, we obtain the conditions under which a family periodic solutions bifurcate from the positive equilibrium E^* at the critical values of $\tau = \tau_j$. As pointed out in Hassard et al. [27], it is interesting to determine the direction, stability and period of these periodic solutions bifurcating. Following the ideas of Hassard et al. we shall establish the explicit formulae determining the properties of the Hopf bifurcation at the critical value of τ using the normal form and the center manifold theory.

Letting $\bar{y}_i(t) = y_i(\tau t)$, $\tau = \tau_j + \mu$ and dropping the bars for simplification of notation, system (3.1) is transformed into an FDE in $C = C([-1, 0], R^3)$ as

$$\dot{y}(t) = L_\mu(y_t) + h(\mu, y_t), \tag{4.1}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^T \in R^3$, and $L_\mu : C \to R^3,\ h : R \times C \to R^3$ are given, respectively, by

$$\begin{aligned} L_\mu\varphi \;=\; & (\tau_j + \mu)\begin{pmatrix} -a & -D^* & -D^* \\ b_{11}x_1^* & 0 & 0 \\ b_{21}x_2^* & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} \\ & +(\tau_j + \mu)\begin{pmatrix} 0 & k_1(1 - S^*) & k_2(1 - S^*) \\ 0 & -k_1x_1^* & -k_2x_1^* \\ 0 & -k_1x_2^* & -k_2x_2^* \end{pmatrix}\begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{pmatrix}, \end{aligned} \tag{4.2}$$

and

$$h(\mu, \varphi) = (\tau_j + \mu)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \tag{4.3}$$

where

$$\begin{aligned} h_1 =& -(b_{12}x_1^* + b_{22}x_2^*)\varphi_1^2(0) - b_{11}\varphi_1(0)\varphi_2(0) - b_{21}\varphi_1(0)\varphi_3(0) - k_1\varphi_1(0)\varphi_2(-1) \\ & - k_2\varphi_1(0)\varphi_3(-1) - (x_1^* b_{13} + x_2^* b_{23})\varphi_1^3(0) - b_{12}\varphi_1^2(0)\varphi_2(0) - b_{22}\varphi_1^2(0)\varphi_3(0) + \cdots, \\ h_2 =& \, b_{12}x_1^*\varphi_1^2(0) + b_{11}\varphi_1(0)\varphi_2(0) - k_1\varphi_2(0)\varphi_2(-1) - k_2\varphi_2(0)\varphi_3(-1) \\ & + b_{13}x_1^*\varphi_1^3(0) + b_{12}\varphi_1^2(0)\varphi_2(0) + \cdots, \\ h_3 =& \, b_{22}x_2^*\varphi_1^2(0) + b_{21}\varphi_1(0)\varphi_3(0) - k_1\varphi_2(-1)\varphi_3(0) - k_2\varphi_3(0)\varphi_3(-1) \\ & + b_{23}x_2^*\varphi_1^3(0) + b_{22}\varphi_1^2(0)\varphi_3(0) + \cdots. \end{aligned}$$

By the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu \varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta) \quad \text{for } \varphi \in C. \tag{4.4}$$

In fact, we can choose

$$\begin{aligned} \eta(\theta, \mu) =& (\tau_j + \mu) \begin{pmatrix} -a & -D^* & -D^* \\ x_1^* b_{11} & 0 & 0 \\ x_2^* b_{21} & 0 & 0 \end{pmatrix} \delta(\theta) \\ & + (\tau_j + \mu) \begin{pmatrix} 0 & k_1(1 - S^*) & k_2(1 - S^*) \\ 0 & -k_1 x_1^* & -k_2 x_1^* \\ 0 & -k_1 x_2^* & -k_2 x_2^* \end{pmatrix} \delta(\theta + 1), \end{aligned} \tag{4.5}$$

where δ is the Dirac delta function.

For $\varphi \in C^1([-1, 0], R^3)$, define

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), & \theta = 0, \end{cases} \tag{4.6}$$

and

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ h(\mu, \varphi), & \theta = 0, \end{cases} \tag{4.7}$$

Then system (4.1) is equivalent to

$$\dot{y}_t = A(\mu)y_t + R(\mu)y_t, \tag{4.8}$$

where $y_t(\theta) = y(t + \theta)$ for $\theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (R^3)^*)$, define

$$A^*\psi = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0 \end{cases} \tag{4.9}$$

and a bilinear inner product

$$\langle \psi(s), \varphi(\theta) \rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \varphi(\xi) d\xi, \tag{4.10}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Sect. 3, we know that $\pm i\omega_* \tau_j$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega_*\tau_j$ and $-i\omega_*\tau_j$, respectively.

Suppose that $q(\theta) = (1, \alpha_1, \beta_1)^T e^{i\omega_*\tau_j\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_*\tau_j$. Then $A(0)q(\theta) = i\omega_*\tau_j q(\theta)$, it follows from the definition of $A(0)$ and $\eta(\theta, \mu)$ that

$$\tau_j \begin{pmatrix} i\omega_* + a & D^* - k_1(1 - S^*)e^{-i\omega_*\tau_j} & D^* - k_2(1 - S^*)e^{-i\omega_*\tau_j} \\ -b_{11}x_1^* & i\omega_* + k_1 x_1^* e^{-i\omega_*\tau_j} & k_2 x_1^* e^{-i\omega_*\tau_j} \\ -b_{21}x_2^* & k_1 x_2^* e^{-i\omega_*\tau_j} & i\omega_* + k_2 x_2^* e^{-i\omega_*\tau_j} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Noting that $d + k_1x_1^* + k_2x_2^* = D^*$ and $S^* + x_1^* + x_2^* = 1$, by direct computation, it is not difficult to show that

$$\alpha_1 = \frac{[i(b_{21} - b_{11})x_2^* - \omega_*]x_1^*}{(x_1^* + x_2^*)\omega_*},$$
$$\beta_1 = \frac{-[i(b_{21} - b_{11})x_1^* + \omega_*]x_2^*}{(x_1^* + x_2^*)\omega_*}. \tag{4.11}$$

Similarly, suppose $q^*(s) = D(1, \alpha_2, \beta_2)e^{i\omega_*\tau_j s}$ is the eigenvector of A^* corresponding to $-i\omega_*\tau_j$. By the definition of A^* and (4.4) and (4.5), we have

$$\tau_j \begin{pmatrix} -i\omega_* + a & -x_1^* b_{11} & -x_2^* b_{21} \\ D^* - k_1(1 - S^*)e^{i\omega_*\tau_j} & -i\omega_* + k_1 x_1^* e^{i\omega_*\tau_j} & k_1 x_2^* e^{i\omega_*\tau_j} \\ D^* - k_2(1 - S^*)e^{i\omega_*\tau_j} & k_2 x_1^* e^{i\omega_*\tau_j} & -i\omega_* + k_2 x_2^* e^{i\omega_*\tau_j} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus we obtain that

$$\alpha_2 = \frac{ak_1\omega_* - i[k_1\omega_*^2 + (k_2 - k_1)b_{21}D^*x_2^*]}{\omega_*(k_1x_1^*b_{11} + k_2x_2^*b_{21})},$$
$$\beta_2 = \frac{ak_2\omega_* - i[k_2\omega_*^2 - (k_2 - k_1)b_{11}D^*x_2^*]}{\omega_*(k_1x_1^*b_{11} + k_2x_2^*b_{21})}. \tag{4.12}$$

In order to assure $\langle q^*(s), q(\theta)\rangle = 1$, we need to determine the value of D. Form (4.10), we have

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \bar{D}(1, \bar{\alpha}_2, \bar{\beta}_2)(1, \alpha_1, \beta_1)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}_2, \bar{\beta}_2)e^{-i\omega_*\tau_j(\xi-\theta)}d\eta(\theta)(1, \alpha_1, \beta_1)^T e^{i\omega_*\tau_j\xi}d\xi \\ &= \bar{D}\left\{1 + \alpha_1\bar{\alpha}_2 + \beta_1\bar{\beta}_2 - \int_{-1}^{0}(1, \bar{\alpha}_2, \bar{\beta}_2)\theta e^{i\omega_*\tau_j\theta}d\eta(\theta)(1, \alpha_1, \beta_1)^T\right\} \\ &= \bar{D}\{1 + \alpha_1\bar{\alpha}_2 + \beta_1\bar{\beta}_2 + \tau_j(k_1\alpha_1 \\ &\quad + k_2\beta_1)(1 - S^* - x_1^*\bar{\alpha}_2 - x_2^*\bar{\beta}_2)e^{-i\omega_*\tau_j}\}. \end{aligned} \tag{4.13}$$

Hence, we can choose D as

$$D = \frac{1}{1 + \bar{\alpha}_1\alpha_2 + \bar{\beta}_1\beta_2 + \tau_j(k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)(1 - S^* - x_1^*\alpha_2 - x_2^*\beta_2)e^{i\omega_*\tau_j}}, \tag{4.14}$$

which assures that $\langle q^*(s), q(\theta)\rangle = 1$.

In the remainder of this section, we will follow the ideas and use the same notations as in Hassard et al. [27]. We first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let y_t be the solution of Eq. (4.1) when $\mu = 0$. Define

$$z(t) = \langle q^*(s), y_t(\theta)\rangle, \quad W(t, \theta) = y_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \tag{4.15}$$

On the center manifold C_0 we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z, \bar{z}, \theta) = \frac{1}{2}W_{20}(\theta)z^2 + W_{11}z\bar{z} + \frac{1}{2}W_{02}\bar{z}^2 + \cdots, \tag{4.16}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if y_t is real. We consider only real solutions. For the solution $y_t \in C_0$ of (4.8), since $\mu = 0$, we have

$$\begin{aligned}\dot{z}(t) &= i\omega_*\tau_j z + \langle \bar{q}^*(\theta), h(0, W(z, \bar{z}, \theta) + 2\mathrm{Re}\{zq(\theta)\})\rangle \\ &= i\omega_*\tau_j z + \bar{q}^*(0)h(0, W(z, \bar{z}, 0) + 2\mathrm{Re}\{zq(0)\}) \\ &\stackrel{\text{def}}{=} i\omega_*\tau_j z + \bar{q}^*(0)h_0(z, \bar{z}). \end{aligned} \tag{4.17}$$

We rewrite this equation as

$$z'(t) = i\omega_* \tau_j z(t) + g(z, \bar{z}), \tag{4.18}$$

where

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) h_0(z, \bar{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots . \end{aligned} \tag{4.19}$$

Noticing $y_t(\theta) = (y_{1t}(\theta), y_{2t}(\theta), y_{3t}(\theta))^T = W(t, \theta) + zq(\theta) + \overline{zq(\theta)}, q(\theta) = (1, \alpha_1, \beta_1)^T e^{i\omega_* \tau_j \theta}$, we have

$$\begin{aligned} y_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ y_{2t}(0) &= \alpha_1 z + \bar{\alpha}_1 \bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ y_{3t}(0) &= \beta_1 z + \bar{\beta}_1 \bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0) z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ y_{2t}(-1) &= \alpha_1 e^{-i\omega_* \tau_j} z + \bar{\alpha}_1 e^{i\omega_* \tau_j} \bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} \\ &\quad + O(|(z, \bar{z})|^3), \\ y_{3t}(-1) &= \beta_1 e^{-i\omega_* \tau_j} z + \bar{\beta}_1 e^{i\omega_* \tau_j} \bar{z} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1) z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} \\ &\quad + O(|(z, \bar{z})|^3). \end{aligned} \tag{4.20}$$

From (4.19), we have

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) h_0(z, \bar{z}) = \bar{D}(1, \bar{\alpha}_2, \bar{\beta}_2)\tau_j (h_1, h_2, h_3)^T \\ &= \bar{D}\tau_j \{-(x_1^* b_{12} + x_2^* b_{22})\varphi_1^2(0) - b_{11}\varphi_1(0)\varphi_2(0) - b_{21}\varphi_1(0)\varphi_3(0) \\ &\quad - k_1 \varphi_1(0)\varphi_2(-1) - k_2\varphi_1(0)\varphi_3(-1) - (x_1^* b_{13} + x_2^* b_{23})\varphi_1^3(0) \\ &\quad - b_{12}\varphi_1^2(0)\varphi_2(0) - b_{22}\varphi_1^2(0)\varphi_3(0) + \bar{\alpha}_2[x_1^* b_{12}\varphi_1^2(0) + b_{11}\varphi_1(0)\varphi_2(0) \\ &\quad - k_1\varphi_2(0)\varphi_2(-1) - k_2\varphi_2(0)\varphi_3(-1) + x_1^* b_{13}\varphi_1^3(0) \\ &\quad + b_{12}\varphi_1^2(0)\varphi_2(0)] + \bar{\beta}_2[x_2^* b_{22}\varphi_1^2(0) + b_{21}\varphi_1(0)\varphi_3(0) - k_1\varphi_2(-1)\varphi_3(0) \\ &\quad - k_2\varphi_3(0)\varphi_3(-1) + x_2^* b_{23}\varphi_1^3(0) + b_{22}\varphi_1^2(0)\varphi_3(0)]\} \\ &= \bar{D}\tau_j \{[-(x_1^* b_{13} + x_2^* b_{23}) + x_1^*\bar{\alpha}_2 b_{13} + x_2^*\bar{\beta}_2 b_{23}]\varphi_1^3(0) + b_{12}(\bar{\alpha}_2 - 1)\varphi_1^2(0)\varphi_2(0) \\ &\quad + b_{22}(\bar{\beta}_2 - 1)\varphi_1^2(0)\varphi_3(0) + [-(x_1^* b_{12} + x_2^* b_{22}) + x_1^*\bar{\alpha}_2 b_{12} + x_2^*\bar{\beta}_2 b_{22}]\varphi_1^2(0) \\ &\quad + b_{11}(\bar{\alpha}_2 - 1)\varphi_1(0)\varphi_2(0) + b_{21}(\bar{\beta}_2 - 1)\varphi_1(0)\varphi_3(0) - k_1\varphi_1(0)\varphi_2(-1) \\ &\quad - k_2\varphi_1(0)\varphi_3(-1) - \bar{\alpha}_2 k_1\varphi_2(0)\varphi_2(-1) - \bar{\alpha}_2 k_2\varphi_2(0)\varphi_3(-1) \\ &\quad - \bar{\beta}_2 k_1\varphi_2(-1)\varphi_3(0) - \bar{\beta}_2 k_2\varphi_3(0)\varphi_3(-1). \end{aligned} \tag{4.21}$$

By substituting (4.20) into (4.21) and comparing the coefficients with (4.19), we get

$$
\begin{aligned}
g_{20} = & -2\bar{D}\tau_j[x_1^*b_{12} + x_2^*b_{22} + b_{11}\alpha_1 + b_{21}\beta_1 + (k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}] \\
& + 2\bar{D}\tau_j\bar{\alpha}_2[x_1^*b_{12} + b_{11}\alpha_1 - \alpha_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}] \\
& + 2\bar{D}\tau_j\bar{\beta}_2[x_2^*b_{22} + b_{21}\beta_1 - \beta_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}], \\
g_{11} = & -\bar{D}\tau_j[2(x_1^*b_{12} + x_2^*b_{22}) + b_{11}(\bar{\alpha}_1 + \alpha_1) + b_{21}(\bar{\beta}_1 + \beta_1) \\
& + (k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)e^{i\omega_*\tau_j} + (k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}] - \bar{D}\tau_j\bar{\alpha}_2[-2x_1^*b_{12} - b_{11}(\bar{\alpha}_1 + \alpha_1) \\
& + \alpha_1(k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)e^{i\omega_*\tau_j} + \bar{\alpha}_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}] - \bar{D}\tau_j\bar{\beta}_2[-2x_2^*b_{22} \\
& - b_{21}(\bar{\beta}_1 + \beta_1) + \beta_1(k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)e^{i\omega_*\tau_j} + \bar{\beta}_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}], \\
g_{02} = & -2\bar{D}\tau_j[x_1^*b_{12} + x_2^*b_{22} + \bar{\alpha}_1 b_{11} + \bar{\beta}_1 b_{21} + (k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)e^{i\omega_*\tau_j}] \\
& + 2\bar{D}\tau_j\bar{\alpha}_2[x_1^*b_{12} + b_{11}\bar{\alpha}_1 - \bar{\alpha}_1(k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)e^{i\omega_*\tau_j}] \\
& + 2\bar{D}\tau_j\bar{\beta}_2[x_2^*b_{22} + b_{21}\bar{\beta}_1 - \bar{\beta}_1(k_1\bar{\alpha}_1 + k_2\bar{\beta}_1)e^{i\omega_*\tau_j}], \\
g_{21} = & -2\bar{D}\tau_j[3(x_1^*b_{13} + x_2^*b_{23}) + b_{12}(\bar{\alpha}_1 + 2\alpha_1) + b_{22}(\bar{\beta}_1 + 2\beta_1) \\
& + (x_1^*b_{12} + x_2^*b_{22})(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + b_{11}M_1 + b_{21}M_2 + k_1M_3 + k_2M_4] \\
& + 2\bar{D}\tau_j\bar{\alpha}_2[3x_1^*b_{13} + b_{12}(\bar{\alpha}_1 + 2\alpha_1) + x_1^*b_{12}(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) \\
& + b_{11}M_1 - k_1M_5 - k_2M_6] + 2\bar{D}\tau_j\bar{\beta}_2[3x_2^*b_{23} + b_{22}(\bar{\beta}_1 + 2\beta_1) \\
& + x_2^*b_{22}(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + b_{21}M_2 - k_1M_7 - k_2M_8],
\end{aligned} \tag{4.22}
$$

where in g_{21},

$$
\begin{aligned}
M_1 &= W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}\bar{\alpha}_1 W_{20}^{(1)}(0) + \alpha_1 W_{11}^{(1)}(0), \\
M_2 &= W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0) + \frac{1}{2}\bar{\beta}_1 W_{20}^{(1)}(0) + \beta_1 W_{11}^{(1)}(0), \\
M_3 &= W_{11}^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1) + \frac{1}{2}\bar{\alpha}_1 W_{20}^{(1)}(0)e^{i\omega_*\tau_j} + \alpha_1 W_{11}^{(1)}(0)e^{-i\omega_*\tau_j}, \\
M_4 &= W_{11}^{(3)}(-1) + \frac{1}{2}W_{20}^{(3)}(-1) + \frac{1}{2}\bar{\beta}_1 W_{20}^{(1)}(0)e^{i\omega_*\tau_j} + \beta_1 W_{11}^{(1)}(0)e^{-i\omega_*\tau_j}, \\
M_5 &= \alpha_1 W_{11}^{(2)}(-1) + \frac{1}{2}\bar{\alpha}_1 W_{20}^{(2)}(-1) + \frac{1}{2}\bar{\alpha}_1 W_{20}^{(2)}(0)e^{i\omega_*\tau_j} + \alpha_1 W_{11}^{(2)}(0)e^{-i\omega_*\tau_j}, \\
M_6 &= \alpha_1 W_{11}^{(3)}(-1) + \frac{1}{2}\bar{\alpha}_1 W_{20}^{(3)}(-1) + \frac{1}{2}\bar{\beta}_1 W_{20}^{(2)}(0)e^{i\omega_*\tau_j} + \beta_1 W_{11}^{(2)}(0)e^{-i\omega_*\tau_j}, \\
M_7 &= \alpha_1 W_{11}^{(3)}(0)e^{-i\omega_*\tau_j} + \frac{1}{2}\bar{\alpha}_1 W_{20}^{(3)}(0)e^{i\omega_*\tau_j} + \frac{1}{2}\bar{\beta}_1 W_{20}^{(2)}(-1) + \beta_1 W_{11}^{(2)}(-1), \\
M_8 &= \beta_1 W_{11}^{(3)}(-1) + \frac{1}{2}\bar{\beta}_1 W_{20}^{(3)}(-1) + \frac{1}{2}\bar{\beta}_1 W_{20}^{(3)}(0)e^{i\omega_*\tau_j} + \beta_1 W_{11}^{(3)}(0)e^{-i\omega_*\tau_j}.
\end{aligned}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From (4.8) and (4.17), we have

$$\begin{aligned}\dot{W} &= \dot{y}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2Re\{\bar{q}^*(0)h_0(z,\bar{z})q(\theta)\}, & \theta \in [-1,0), \\ AW - 2Re\{\bar{q}^*(0)h_0(z,\bar{z})q(0)\} + h_0(z,\bar{z}), & \theta = 0, \end{cases} \\ &\stackrel{\text{def}}{=} AW + H(z,\bar{z},\theta),\end{aligned} \tag{4.23}$$

where

$$H(z,\bar{z},\theta) = \frac{1}{2}H_{20}(\theta)z^2 + H_{11}(\theta)z\bar{z} + \frac{1}{2}H_{02}(\theta)\bar{z}^2 + \frac{1}{6}H_{20}(\theta)z^3 + \cdots. \tag{4.24}$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$(A - 2i\omega_*\tau_j I)W_{20}(\theta) = -H_{20}(\theta),\ AW_{11}(\theta) = -H_{11}(\theta), \ldots. \tag{4.25}$$

From (4.23), we know that for $\theta \in [-1, 0)$,

$$\begin{aligned}H(z,\bar{z},\theta) &= -\bar{q}^*(0)h_0(z,\bar{z})q(\theta) - q^*(0)\bar{h}_0(z,\bar{z})\bar{q}(\theta) \\ &= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \\ &= \frac{1}{2}(g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta))z^2 - (g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta))z\bar{z} + \cdots.\end{aligned} \tag{4.26}$$

Comparing the coefficients with (4.24) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta),\ H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{4.27}$$

From (4.25), (4.27) and the definition of A, it follows that

$$\dot{W}_{20}(\theta) = 2i\omega_*\tau_j W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Notice that $q(\theta) = (1, \alpha_1, \beta_1)^T e^{i\omega_*\tau_j\theta}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_*\tau_j}q(0)e^{i\omega_*\tau_j\theta} + \frac{i\bar{g}_{02}}{3\omega_*\tau_j}\bar{q}(0)e^{-i\omega_*\tau_j\theta} + E_1 e^{2i\omega_*\tau_j\theta}, \tag{4.28}$$

where $E_1 = (E_{11}, E_{12}, E_{13})^T \in R^3$ is a constant vector.

Similarly, from (4.25) and (4.27), we can obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_*\tau_j}q(0)e^{i\omega_*\tau_j\theta} + \frac{i\bar{g}_{11}}{\omega_*\tau_j}\bar{q}(0)e^{-i\omega_*\tau_j\theta} + E_2, \tag{4.29}$$

where $E_2 = (E_{21}, E_{22}, E_{23})^T \in R^3$ is also a constant vector.

In what follows, we shall seek appropriate E_1 and E_2. From the definition of A and (4.25), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_*\tau_j W_{20}(0) - H_{20}(0),\ \int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{4.30}$$

where $\eta(\theta) = \eta(0, \theta)$.

Note that $q(\theta)$ is the eigenvector of $A(0)$ and from (4.28) and the definition of $A(0)$, we know that

$$\begin{aligned}\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) &= \frac{ig_{20}}{\omega_*\tau_j}\int_{-1}^{0} d\eta(\theta)q(\theta)\\ &\quad+\frac{i\bar{g}_{02}}{3\omega_*\tau_j}\int_{-1}^{0} d\eta(\theta)\bar{q}(\theta)+\int_{-1}^{0} d\eta(\theta)E_1e^{2i\omega_*\tau_j\theta}\\ &= \frac{ig_{20}}{\omega_*\tau_j}(i\omega_*\tau_j q(0))+\frac{i\bar{g}_{02}}{3\omega_*\tau_j}(-i\omega_*\tau_j\bar{q}(0))\\ &\quad+\int_{-1}^{0} d\eta(\theta)E_1e^{2i\omega_*\tau_j\theta}\\ &= -g_{20}q(0)+\frac{\bar{g}_{02}}{3}\bar{q}(0)+\int_{-1}^{0} d\eta(\theta)E_1e^{2i\omega_*\tau_j\theta} \end{aligned} \tag{4.31}$$

and

$$2i\omega_*\tau_j W_{20}(0) = -2g_{20}q(0) - 2\frac{\bar{g}_{02}}{3}\bar{q}(0) + 2i\omega_*\tau_j E_1. \tag{4.32}$$

Thus, the first equation of (4.30) becomes

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \left[2i\omega_*\tau_j - \int_{-1}^{0} d\eta(\theta)e^{2i\omega_*\tau_j\theta}\right]E_1. \tag{4.33}$$

Similarly, from (4.29), we have

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = g_{11}q(0) + \bar{g}_{11}\bar{q}(0) + \int_{-1}^{0} d\eta(\theta)E_2. \tag{4.34}$$

Hence, the second equation of (4.30) becomes

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - \int_{-1}^{0} d\eta(\theta)E_2. \tag{4.35}$$

From (4.23), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_j(w_1, w_2, w_3)^T, \tag{4.36}$$

where

$$\begin{aligned}
w_1 &= -(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j} - (x_1^*b_{12} + b_{11}\alpha_1 + x_2^*b_{22} + b_{21}\beta_1),\\
w_2 &= x_1^*b_{12} + b_{11}\alpha_1 - \alpha_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j},\\
w_3 &= x_2^*b_{22} + b_{21}\beta_1 - \beta_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}
\end{aligned}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - 2\tau_j(v_1, v_2, v_3)^T, \tag{4.37}$$

where

$$\begin{aligned}
v_1 &= Re\{b_{11}\alpha_1 + b_{21}\beta_1 + (k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j}\} + (x_1^*b_{12} + x_2^*b_{22}),\\
v_2 &= Re\{\bar{\alpha}_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j} - b_{11}\alpha_1\} - x_1^*b_{12},\\
v_3 &= \mathrm{Re}\{\bar{\beta}_1(k_1\alpha_1 + k_2\beta_1)e^{-i\omega_*\tau_j} - b_{21}\beta_1\} - x_2^*b_{22}.
\end{aligned}$$

Substituting (4.36) into (4.33), one get

$$\left[2i\omega_*\tau_j - \int_{-1}^{0} d\eta(\theta)e^{2i\omega_*\tau_j\theta}\right]E_1 = 2\tau_j(w_1, w_2, w_3)^T,$$

that is

$$\begin{pmatrix} 2\omega_*i + a & D^* - k_1(1-S^*)e^{-2i\omega_*\tau_j} & D^* - k_2(1-S^*)e^{-2i\omega_*\tau_j} \\ -x_1^*b_{11} & 2\omega_*i + k_1x_1^*e^{-2i\omega_*\tau_j} & k_2x_1^*e^{-2i\omega_*\tau_j} \\ -x_2^*b_{21} & k_1x_2^*e^{-2i\omega_*\tau_j} & 2\omega_*i + k_2x_2^*e^{-2i\omega_*\tau_j} \end{pmatrix} E_1 = 2\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

It follows that

$$E_1 = 2\begin{pmatrix} 2\omega_*i + a & D^* - k_1(1-S^*)e^{-2i\omega_*\tau_j} & D^* - k_2(1-S^*)e^{-2i\omega_*\tau_j} \\ -x_1^*b_{11} & 2\omega_*i + k_1x_1^*e^{-2i\omega_*\tau_j} & k_2x_1^*e^{-2i\omega_*\tau_j} \\ -x_2^*b_{21} & k_1x_2^*e^{-2i\omega_*\tau_j} & 2\omega_*i + k_2x_2^*e^{-2i\omega_*\tau_j} \end{pmatrix}^{-1} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}. \tag{4.38}$$

Substituting (4.37) into (4.35), one get

$$\begin{pmatrix} a & D^* - k_1(1-S^*) & D^* - k_2(1-S^*) \\ -x_1^*b_{11} & k_1x_1^* & k_2x_1^* \\ -x_2^*b_{21} & k_1x_2^* & k_2x_2^* \end{pmatrix} E_2 = -2\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

It follows that

$$E_2 = -2\begin{pmatrix} a & D^* - k_1(1-S^*) & D^* - k_2(1-S^*) \\ -x_1^*b_{11} & k_1x_1^* & k_2x_1^* \\ -x_2^*b_{21} & k_1x_2^* & k_2x_2^* \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{4.39}$$

Thus, we can determine $W_{20}(0)$ and $W_{11}(0)$ from (4.29) and (4.30). Furthermore, we can determine g_{21}. Therefore, each g_{ij} in (4.19) is determined by the parameters and delay in (4.22). Thus, we can compute the following values:

$$c_1(0) = \frac{i}{2\omega_* \tau_j}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau_j))}, \tag{4.40}$$
$$\mathscr{B}_2 = 2Re(c_1(0)),$$
$$T_2 = -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_j))}{\omega_* \tau_j}, \tag{4.41}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_j, i.e., μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical(subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_j$ ($\tau < \tau_j$); $\mathscr{B}_2$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable(unstable) if $\mathscr{B}_2 < 0$ ($\mathscr{B}_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0$ ($T_2 < 0$).

5 A numerical example

In this section, we give a numerical simulation supporting the theoretical analysis. We consider the following system

$$\begin{aligned}
\dot{S} &= (1.4 + 2.2x_1(t-\tau) + 1.05x_2(t-\tau))(1-S) - \frac{3S}{0.25+S}x_1 - \frac{5S}{0.8+S}x_2,\\
\dot{x}_1 &= x_1\left[\frac{3S}{0.25+S} - (1.4 + 2.2x_1(t-\tau) + 1.05x_2(t-\tau))\right],\\
\dot{x}_2 &= x_2\left[\frac{5S}{0.8+S} - (1.4 + 2.2x_1(t-\tau) + 1.05x_2(t-\tau))\right],
\end{aligned} \tag{5.1}$$

where $k_1 = 2.2, k_2 = 1.05, d = 1.4$ and $f_1(S) = \frac{3S}{0.25+S},\ f_2(S) = \frac{5S}{0.8+S}$. The positive equilibrium is $E_* \doteq (0.5750, 0.2127, 0.2123)$.

Simple calculations show that $q \doteq 0.5264e^{-1} > 0$, thus, the conditions in Theorem 3.1 (2) is satisfied. Furthermore, we can compute from (3.8)–(3.10) that $\omega_* \doteq 0.2408$, $\tau_0 \doteq 10.3671$ and $\frac{dRe\{\lambda_0(\tau)\}}{d\tau}|_{\tau=\tau_0} \doteq 0.1824e^{-2} > 0$. By Theorem 3.1, the positive equilibrium E^* is asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$ (see Fig. 1a–d), where $\tau = 10$). When τ passes through the critical value τ_0 to the right, E^* loses its stability forever and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium E^* (see Fig. 2a–d), where $\tau = 13$).

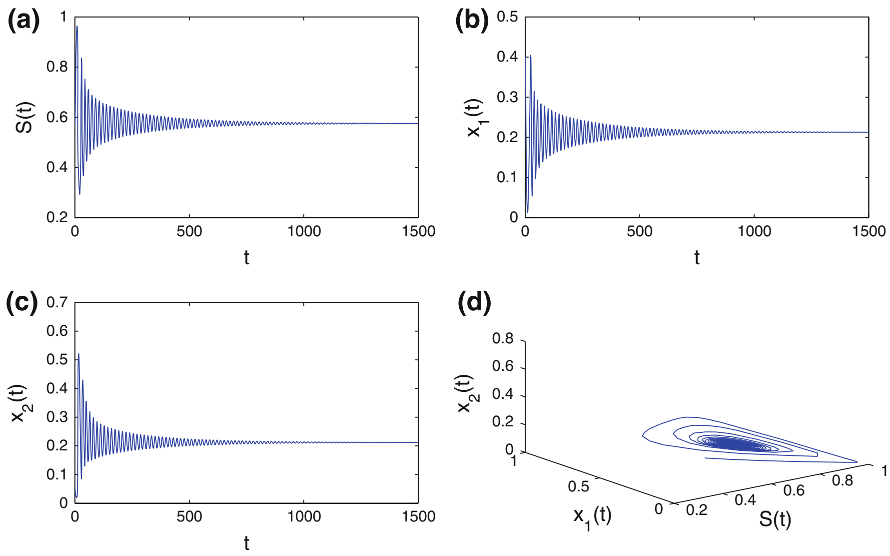

Fig. 1 The positive equilibrium $E^* \doteq (0.5750, 0.2127, 0.2123)$ of (5.1) is asymptotically stable when $\tau < \tau_0$. **a** Time-state $t - S(t)$ relation graph. **b** Time-state $t - x_1(t)$ relation graph. **c** Time-state $t - x_2(t)$ relation graph. **d** Phase portrait of system (5.1). Here $(S(0), x_1(0), x_2(0)) = (0.57, 0.4, 0.05)$ and $\tau = 10$

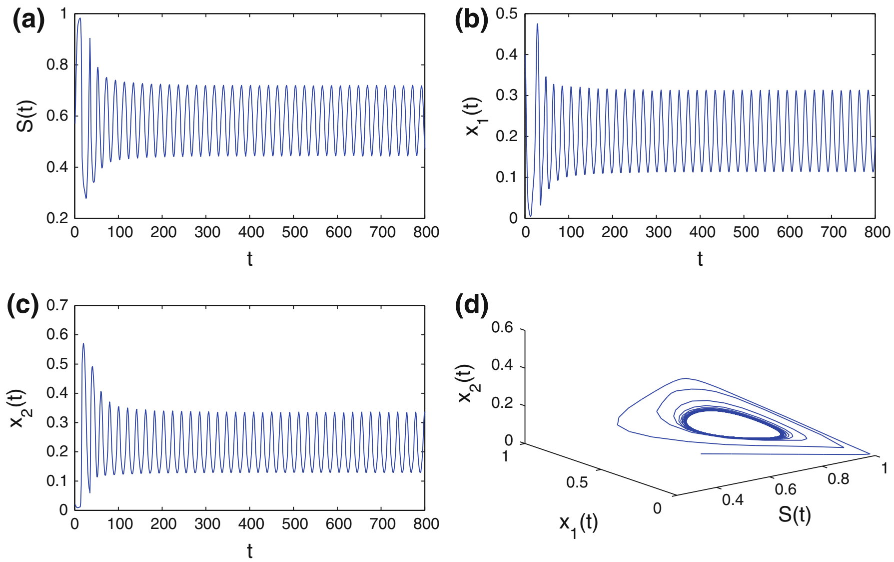

Fig. 2 The positive equilibrium $E^* \doteq (0.5750, 0.2127, 0.2123)$ of (5.1) is unstable when $\tau > \tau_0$ and a bifurcating stable periodic solution from E^* occurs. **a** Time-state $t - S(t)$ relation graph. **b** Time-state $t - x_1(t)$ relation graph. **c** Time-state $t - x_2(t)$ relation graph. **d** Phase portrait of system (5.1). Here $(S(0), x_1(0), x_2(0)) = (0.57, 0.4, 0.02)$ and $\tau = 13$

The properties of bifurcating periodic solutions can be obtained by the results of Sect. 4. According to (4.22), we can get

$$\begin{aligned} g_{20} &\doteq 0.4260 + 1.9808i, \\ g_{11} &\doteq 1.5627 - 0.3156i, \\ g_{02} &\doteq 0.5892 + 2.5785i, \\ g_{21} &\doteq -11.5342 + 7.5949i. \end{aligned}$$

Hence, Eq. (4.40) can be calculated exactly as follows:

$$\begin{aligned} c_1(0) &\doteq -6.3602 + 2.5708i, \quad \mu_2 \doteq 3486.1063 > 0, \\ \mathscr{B}_2 &\doteq -12.7203 < 0, \quad T_2 \doteq 30.8712 > 0. \end{aligned}$$

Therefore, we can conclude that the Hopf bifurcation for system (5.1) is supercritical for $\tau > \tau_0$ and the bifurcating periodic solution is asymptotically stable since $\mu_2 > 0$ and $\mathscr{B}_2 < 0$.

6 Discussion

It was shown in [18] that a chemostat with two organisms can be made coexistent by means of feedback control of the dilution rate, i.e., if the dilution rate depends affinely on the concentrations of two competing organisms, coexistence may be achieved as a globally asymptotically stable equilibrium point in the interior of the non-negative orthant (see also [15,17–21] for some pertinent studies). Note that it is assumed in all the researches above that the optical sensor used to measure the turbidity of the fluid is fast and accurate. Thus the time needed to the measurement of the optical sensor to the turbidity of the fluid is neglected. But in reality, no matter how sensitive the sensor is there always exists delay in the process of its measurement to the the turbidity of the fluid and this signal is used to control the dilution rate. In this paper, by choosing this as a bifurcation parameter, we show that Hopf bifurcations can occur as the delay crosses some critical values, i.e., a family of periodic solutions will be bifurcated from the positive equilibrium. The direction and stability of the bifurcating periodic solutions are determined by the normal form theory and the center manifold theorem. Computer simulations illustrate the results.

As pointed above, when $\tau = 0$ in model (2.1), the positive equilibrium is globally asymptotically stable (Lemma 2.1, see also [18] for the details). We guess it is globally asymptotically stable for $\tau \in [0, \tau_0)$, too. We leave this for future consideration.

Acknowledgments We would be grateful to Professor Bingtuan Li for sending us his previous works and valuable discussions on this respect. Research is supported by the National Natural Science Foundation of China (10871129), the Educational Committee Foundation of Shanghai (09YZ208) and the Shanghai Committee of Science and Technology (No.10ZR1431900).

References

1. T. Egli, in *The ecology and physiological significance of the growth of heterotrophic microorganisms with mixtures of substrates*, ed. by G Jones Advances in Microbial Ecology, vol. 14 (Plenum Press, New York, 1995), pp. 305–386

2. D. Herbert, R. Elsworth, R.C. Telling, The continuous culture of bacteria a theoretical and experimental study. J. Gen. Microbiol. **14**, 60l–622 (1956)
3. A. Novick, L. Sziliard, Description of the chemostat. Science **112**, 715–716 (1950)
4. H. Smith, P. Waltman, *The Theory of the Chemostat* (Cambridge University Press, Cambridge, UK, 1995)
5. P.A. Taylor, J.L. Williams, Theoretical studies on the coexistence of competing species under continuous-flow conditions. Can. J. Microbiol. **21**, 90–98 (1975)
6. R.A. Armstrong, R. McGehee, Competitive exclusion. Am. Natur. **115**, 151–170 (1980)
7. G.J. Butler, G.S.K. Wolkowicz, A mathematical model of the chemostat with a general class of functions describing nutrient uptake. SIAM J. Appl. Math. **45**, 138–151 (1985)
8. S.B. Hsu, Limiting behavior for competing species. SIAM J. Appl. Math. **34**, 760–763 (1978)
9. S.B. Hsu, S. Hubbell, P. Waltman, A mathematical theory of single-nutrient competition in continuous cultures of micro-organisms. SIAM J. Appl. Math. **32**, 366–383 (1977)
10. B. Li, Global asymptotic behaviour of the chemostat: general response functions and different removal rates. SIAM J. Appl. Math. **59**, 411–422 (1999)
11. G.S.K. Wolkowicz, Z. Lu, Global dynamics of a mathematical model of competition in the chemostat: general response function and differential death rates, SIAM J. Appl. Math., **52**, 222–233 (1992)
12. B. Li (1998) *Analysis of Chemostat-Related Models With Distinct Removal Rates*. Ph.D thesis, Arizona State University
13. G.S.K. Wolkowicz, H. Xia, Global asymptotic behavior of a chemostat model with discrete delays. SIAM J. Appl. Math. **57**, 1019–1043 (1997)
14. G.S.K. Wolkowicz, H. Xia, S. Ruan, Competition in the chemostat:a distributed delay model and its global asymptotic behavior. SIAM J. Appl. Math. **57**, 1281–1310 (1997)
15. J.P. Grover, *Resource Competition* (Chapman Hal, London, 1997)
16. S.R. Hansen, S.P. Hubbell, Single-nutrient microbial competition: agreement between experimental and theoretical forecast outcomes. Science **207**, 1491–1493 (1980)
17. D. Tilman, *Resource Competition and Community Structure* (Princeton U.P., Princeton, 1982)
18. P. De Leenheer, H.L. Smith, Feedback control for the chemostat. J. Math. Biol. **46**, 48–70 (2003)
19. J. Flegr, Two distinct types of natural selection in turbidostat-like and chemostat-like ecosystems. J. Theor. Biol. **188**, 121–126 (1997)
20. B. Li, Competition in a turbidostat for an inhibitory nutrient. J. Biol. Dyn. **2**, 208–220 (2008)
21. S. Yuan, W. Zhang, M. Han, Global asymptotic behavior in chemostat-type competition models with delay. Nonlinear Anal. Real World Appl. **10**, 1305–1320 (2009)
22. N.S Panikov, *Microbial Growth Kinetics* (Chapman Hall, New York, 1995)
23. M.L. Shuler, F. Kargi, *Bioprocess Engineering Basic Concepts* (Prentice Hall, Englewood Cliffs, 1992)
24. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics* (Kluwer Academic Publishers, Boston, 1992)
25. J. Hale, S. Lunel, *Introduction to Functional Differential Equations* (Springer, New York, 1993)
26. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics* (Academic Press, Boston, 1993)
27. B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, *Theory and Applications of Hopf Bifurcation* (Cambridge University Press, Cambridge, 1981)
28. Z. Zhao, T. Wang, L. Chen, Dynamic analysis of a turbidostat model with the feedback control. Comm. Nonlinear Sci. Numer. Simul. **15**, 1028–1035 (2010)
29. G.J. Bulter, S.B. Hsu, P. Waltman, A mathematical model of the chemostat with periodic washout rate. SIAM J. Appl. Math. **45**, 435–449 (1985)
30. S.B. Hsu, A competition model for a seasonally fluctuating nutrient. J. Math. Biol. **9**, 115–132 (1980)
31. H.L. Smith, *Monotone Dynamical System* (AMS, Providence, 1995)
32. J.K. Hale, A.S. Somolinas, Competition for fluctuating nutrient. J. Math. Biol. **18**, 255–280 (1983)
33. G. Stephanopoulos, A.G. Fredrickson, Effect of spatial inhomogeneities on the coexistence of competing microbial populations. Biotech. Bioeng. **XXI**, 1491–1498 (1979)
34. R.S. Cantrell, C. Cosner, S. Ruan, Intraspecific interference and consumer-resource dynamics. Discret. Contin. Dyn. Syst. **4**, 527–546 (2004)
35. X. Meng, Z. Li, J.J. Nieto, Dynamic analysis of Michaelis-Menten chemostat-type competition models with time delay and pulse in a polluted environment. J. Math. Chem. **47**, 123–144 (2010)
36. Z. Zhao, L. Yang, L. Chen, Impulsive state feedback control of the microorganism culture in a turbidostat. J. Math. Chem. **47**, 1224–1239 (2010)

37. H. Zhang, P. Georgescu, J.J. Nieto, L. Chen, Impulsive perturbation and bifurcation of solutions for a model of chemostat with variable yield. Appl. Math. Mech. (English Ed.) **30**, 933–944 (2009)
38. G. Stephanopoulos, A.G. Fredrickson, R. Aris, The growth of competing microbial populations in CSTR with periodically varying inputs. Am. Inst. Chem. Eng. J. **25**, 863–872 (1979)